Constructing Out-of-the-Money Longevity Hedges Using Parametric Mortality Indexes

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Outline

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Mortality Indexes

Purposes of mortality indexes:

- To summarize levels of mortality at different time points
- To quantify mortality and longevity risks
- To construct standardized mortality-linked securities

Advantages of index-based longevity risk transfers:

- More transparent, potentially cheaper and more conductive to liquidity
- Opens up an additional pool of buyers, e.g., hedge funds

Existing Mortality Indexes

- **Credit Suisse**'s longevity index:

  Tracks the life expectancy at birth of the general US population

- QxX Index by **Goldman Sachs**:

  Tracks the number of survivors in the underlying reference pool

- LifeMetrics Indexes by **JP Morgan/LLMA**:

  Tracks the graduated death probabilities of various national populations

- Xpect Cohort Indexes by **Deutsche Börse**:

  Tracks the number of survivors of a certain birth cohort

The Need for Model-Based Mortality Indexes

- A non-parametric mortality index conveys only a limited amount of information.
- To build an effective longevity hedge, a large number of non-parametric indexes are needed.
- **Parametric (model-based) mortality indexes**:
  - Mortality indexes developed from the time-varying parameters in a stochastic mortality model
  - Richer in information content

The CBD Mortality Indexes

- Developed from the Cairns-Blake-Dowd (CBD) model:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}),$$

  where
  - $q_{x,t}$ is the probability that an individual aged x at time t will die between t and $t+1$;
  - $\bar{x}$ is the mean age over the sample age range; and
  - $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are time-varying parameters.

- The time-t values of the first and second **CBD mortality indexes** are defined as $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, respectively.

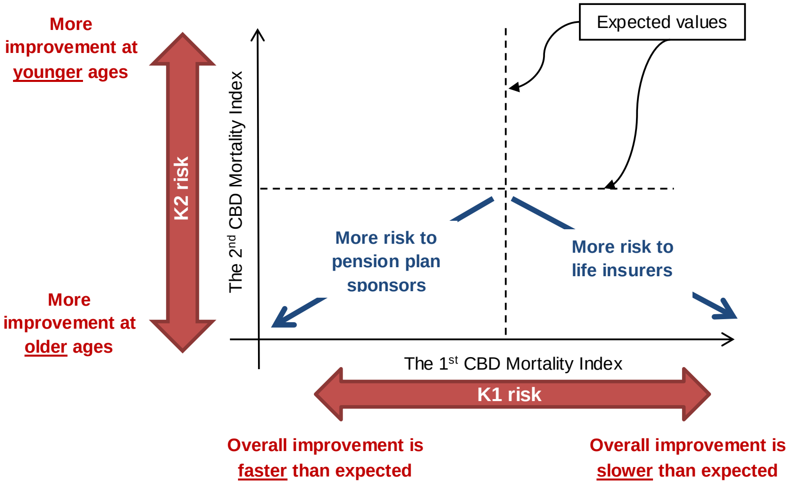
Interpreting the CBD Mortality Indexes

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}),$$

- **The first CBD index** ($\kappa_t^{(1)}$):
  - Represents the level of the mortality curve (after transformation)
  - A reduction in $\kappa_t^{(1)}$ means an overall mortality improvement.
- **The second CBD index** ($\kappa_t^{(2)}$):
  - Represents the slope of the logit-transformed mortality curve
  - An increase in $\kappa_t^{(2)}$ means that mortality at younger ages improves more rapidly than that at older ages

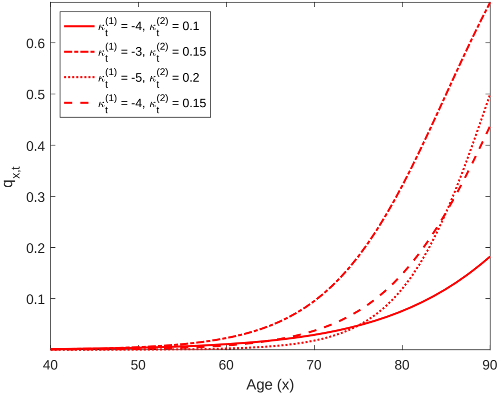
K1 and K2 Risks

Chan et al. (2014) defined **K1 and K2 risks** as the risks surrounding the future values of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, respectively:



Desirable Properties of the CBD Mortality Indexes

- Small in dimension, but able to represent the age-pattern of mortality improvement



- Interpretable
- The <u>new-data-invariant</u> property

Previous Work on Parametric Mortality Indexes

- Chan et al. (2014) introduced parametric mortality indexes and K-forwards.
- Tan et al. (2014) examined how a static K-forward hedge may be calibrated with a 'duration-matching' approach.
- Hao et al. (2017) and Wei (2017) studied the counterparty credit risk associated with K-forwards.
- Biffis et al. (2017) introduced a security that is written on the population-specific time-varying parameters in the augmented common factor (ACF) model.
  - Although they name their security slightly differently ('k-forward' instead of 'K-forward'), the spirit behind is essentially the same.

Objectives of This Research

1. To explore security structures other than a zero coupon swap, and examine how the alternative security structures may benefit the hedger

2. To study risk-neutral valuation of K-forwards and other securities written on parametric mortality indexes

3. To develop static/dynamic hedging strategies based on K-forwards and options

Outline

K-forward

- Consider a K-forward written on the ith CBD mortality index, for $i = 1, 2$.
- Suppose that the K-forward is issued at time t_0 and matures at time T (where $T > t_0$).
- From the perspective of the fixed rate receiver, the payoff of this K-forward at maturity is

$$\mathcal{F}^{(i)}(T, K) = K - \kappa_T^{(i)}$$

per $1 notional, where K represents the fixed leg that is predetermined when the K-forward is issued.

K-call

- Consider a (European) K-call on the ith CBD mortality index, $i = 1, 2$.
- Suppose that the K-call is issued at time t_0, matures at time T, and has a strike value of K.
- The payoff of this K-call at maturity is

$$\mathcal{C}^{(i)}(T, K) = \max(\kappa_T^{(i)} - K, 0)$$

per $1 notional.

K-put

- Consider a (European) K-put on the ith CBD mortality index, $i = 1, 2$.
- Suppose that the K-put is issued at time t_0, matures at time T, and has a strike value of K.
- The payoff of this K-put at maturity is

$$\mathcal{P}^{(i)}(T, K) = \max(K - \kappa_T^{(i)}, 0)$$

per $1 notional.

Hedging for Pension Plans

Liability is higher when...	*Corresponding index pattern*
Overall mortality improvement is faster than expected	Future values of $\kappa_t^{(1)}$ are lower than expected
Mortality improvement is more concentrated at older ages	Future values of $\kappa_t^{(2)}$ are lower than expected

- Pension plans are subject to **downside** K1 and K2 risks.
- To hedge their longevity risk exposures, pension plans may
  - write K1- and K2-forwards as a **fixed-rate receiver**,
  - take a **short position** in K1- and K2-calls, or
  - take a **long position** in K1- and K2-puts.

Hedging for Life Insurance Portfolios

Liability is higher when...	*Corresponding index pattern*
Overall mortality improvement is slower than expected	Future values of $\kappa_t^{(1)}$ are higher than expected
Mortality improvement is more concentrated at older ages	Future values of $\kappa_t^{(2)}$ are lower than expected

- Life insurance portfolios are subject to **upside** K1 risk and **downside** K2 risk.
- To hedge their mortality risk exposures, life insurers may
  - write a K1-forward as a **fixed-rate payer** and a K2-forward as a **fixed-rate receiver**,
  - take a **long position** in a K1-call and a **short position** in a K2-call, or
  - take a **short position** in a K1-put and a **long position** in a K2-put.

Outline

The Real-world and Risk-neutral Processes

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x})$$

- Let $\boldsymbol{\kappa}_t = (\kappa_t^{(1)}, \kappa_t^{(2)})'$.
- Under the real-world probability measure,

$$\boldsymbol{\kappa}_t = \boldsymbol{\mu} + \boldsymbol{\kappa}_{t-1} + \mathbf{A}\boldsymbol{z}_t,$$

where

  - $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})'$ is the constant drift vector,
  - $\mathbf{A}$ is a 2-by-2 upper-triangular matrix, and
  - $\boldsymbol{z}_t = (z_t^{(1)}, z_t^{(2)})'$ is a vector of two uncorrelated standard normal random variables under the real-world probability measure.

The Real-world and Risk-neutral Processes

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x})$$

- Under the risk-neutral probability measure,

$$\boldsymbol{\kappa}_t = \boldsymbol{\mu} + \boldsymbol{\kappa}_{t-1} + \mathbf{A}(\tilde{\boldsymbol{z}}_t - \boldsymbol{\lambda}),$$

or equivalently

$$\boldsymbol{\kappa}_t = \tilde{\boldsymbol{\mu}} + \boldsymbol{\kappa}_{t-1} + \mathbf{A}\tilde{\boldsymbol{z}}_t,$$

where

  - $\tilde{\boldsymbol{z}}_t = (\tilde{z}_t^{(1)}, \tilde{z}_t^{(2)})'$ is a vector of two uncorrelated standard normal random variables under the risk-neutral measure,
  - $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})'$ is the vector of market prices of risk, and
  - $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{A}\boldsymbol{\lambda}$.

The Market Prices of Risk

- The market prices of risk are calibrated to market information, e.g., market prices of individual life annuities.
- In our baseline calculations, we set $\lambda^{(1)} = \lambda^{(2)} = 0.175$.
- This collection of market prices of risk was obtained by Cairns et al. (2006) using the market price of the longevity bond jointly announced by BNP Paribas and the European Investment Bank in 2004.
- We have sensitivity tested the hedging results using a wide range of market prices of risk.

Two Key Quantities

- Given information up to and including time t,

$$\kappa_T^{(i)} = \kappa_t^{(i)} + (T-t)\tilde{\mu}^{(i)} + \sum_{k=1}^{T-t} \tilde{\epsilon}_{t+k}^{(i)}, \quad i = 1, 2,$$

  for $T > t$ under the risk-neutral probability measure, where $\tilde{\epsilon}_{t+k}^{(1)} = a_{1,1}\tilde{z}_{t+k}^{(1)} + a_{1,2}\tilde{z}_{t+k}^{(2)}$, $\tilde{\epsilon}_{t+k}^{(2)} = a_{2,2}\tilde{z}_{t+k}^{(2)}$, and $a_{i,j}$ is the (i,j)th element in $\mathbf{A}$.

- Hence, $\kappa_T^{(i)}$ for $T > t$ under the risk-neutral measure is normally distributed with a mean of

$$\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}] = \kappa_t^{(i)} + (T-t)\tilde{\mu}^{(i)}, \quad i = 1, 2,$$

  and a variance of

$$\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}] = \begin{cases} (T-t)(a_{1,1}^2 + a_{1,2}^2), & i = 1 \\ (T-t)a_{2,2}^2, & i = 2 \end{cases}.$$

Pricing Formula for K-forwards

- For $t_0 \leq t < T$, the time-t value (from the fixed-rate receiver's perspective) of a K-forward with a fixed leg K is

$$\begin{aligned} F_t^{(i)}(T, K) &= \mathrm{E}_t^{(Q)}[(1 + r_f)^{-(T-t)} \mathcal{F}^{(i)}(T, K)] \\ &= (1 + r_f)^{-(T-t)}(K - \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]) \end{aligned}$$

  per \$1 notional, where r_f is the risk-free interest rate.

- In practice, a K-forward may be constructed in such a way that no cash flow exchanged hands when it is issued. To achieve this, we may set the fixed leg K to $\mathrm{E}_{t_0}^{(Q)}[\kappa_T^{(i)}]$.

Longevity Greeks for K-forwards

- Per \$1 notional, the time-t value of the longevity delta of a K-forward is

$$\Delta_t^{(i,F)}(T,K) = \frac{\partial}{\partial \kappa_t^{(i)}} F_t^{(i)}(T,K) = -(1+r)^{-(T-t)},$$

for $i = 1, 2$ and $t_0 \leq t < T$.

- Per \$1 notional, the time-t value of the longevity gamma of a K-forward is

$$\Gamma_t^{(i,F)}(T,K) = \frac{\partial}{\partial \kappa_t^{(i)}} \Delta_t^{(i,F)}(T,K) = 0,$$

for $i = 1, 2$ and $t_0 \leq t < T$.

Pricing Formula for K-calls

Per \$1 notional, and the time-t price of a K-call with a strike K is

$$
\begin{aligned}
C_t^{(i)}(T,K) &= \mathrm{E}_t^{(Q)}[(1+r_f)^{-(T-t)}\mathcal{C}^{(i)}(T,K)] \\
&= (1+r_f)^{-(T-t)}\left(\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}\,\phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)\right. \\
&\quad \left. -\,(K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}])\left(1-\Phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)\right)\right),
\end{aligned}
$$

for $i=1,2$ and $t_0 \le t < T$.

Longevity Greeks for K-calls

Per \$1 notional, the time-t values of the longevity delta and gamma for a K-call with a strike K are

$$\Delta_t^{(i,C)}(T,K) = (1+r_f)^{-(T-t)}\left(1-\Phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)\right)$$

and

$$\Gamma_t^{(i,C)}(T,K) = (1+r_f)^{-(T-t)}\frac{1}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right),$$

for $i=1,2$ and $t_0 \le t < T$, respectively.

Pricing Formula for K-puts

Per \$1 notional, and the time-t price of a K-put with a strike K is

$$\begin{aligned} P_t^{(i)}(T,K) &= \mathrm{E}_t^{(Q)}[(1+r_f)^{-(T-t)}\mathcal{P}^{(i)}(T,K)] \\ &= (1+r_f)^{-(T-t)}\left(\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}\,\phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)\right. \\ &\qquad \left. + \left(K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]\right)\Phi\left(\frac{K-\mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)\right), \end{aligned}$$

for $i=1,2$ and $t_0 \le t < T$.

Longevity Greeks for K-puts

Per \$1 notional, the time-t values of the longevity delta and gamma of a K-put with a strike K are given by

$$\Delta_t^{(i,P)}(T,K) = -(1+r_f)^{-(T-t)}\Phi\left(\frac{K - \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right)$$

and

$$\Gamma_t^{(i,P)}(T,K) = (1+r_f)^{-(T-t)}\frac{1}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\phi\left(\frac{K - \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}\right),$$

for $i = 1, 2$ and $t_0 \leq t < T$, respectively.

Outline

The Hedging Strategy

- At any time, the hedge contains two instruments, one written on each of the two CBD mortality indexes.
- All longevity hedges are established at time t_0.
- Simple delta hedging is considered.
- The notional amount of the *i*th instrument is given by

$$h_t^{(i)} = \frac{\Delta_t^{(i,L)}}{\Delta_t^{(i,H)}(T_t^{(i)}, K_t^{(i)})}$$

where $H = F, C, \text{or } P$ depending on whether the *i*th instrument is a K-forward, K-call or K-put.

Types of Longevity Hedge

- **Cash-flow hedges**
  - Focus on the variability of the cash flows arising from the liability being hedged and the hedging instruments.
  - Static cash-flow hedges: need only $h_t^{(i)}$ for $t = t_0$
  - Dynamic cash-flow hedges: $h_t^{(i)}$ for $t = t_0, \ldots$ are computed.
- **Value hedges**
  - Focus on the variability of the values of the hedged position at a certain future time point, say τ years from time t_0.
  - In our empirical work, we consider particularly $\tau = 1$, which is the most relevant to typical capital requirements.

The Liability Being Hedged

- The liability being hedged is a whole life annuity-immediate of \$1 that is sold to individuals aged x_0 at time t_0.
- The time-t value of the (unpaid) annuity liability is

$$L_t = \sum_{s=1}^{\omega - x_0 - t + t_0} (1 + r_f)^{-s} \mathrm{E}_t^{(Q)}[\mathcal{S}_{x_0+t-t_0,t}(s)],$$

  for $t = t_0, t_0 + 1, \ldots, t_0 + \omega - x_0 - 1$, where $\mathcal{S}_{x_0+t-t_0,t}(s)$ represents the *ex post* probability that an individual aged $x_0 + t - t_0$ at time t would have survived to time $t + s$.
- There is no analytical formula for L_t.

Computational Challenges

- In the following situations, nested simulations are required to estimate the exact values of L_t for $t > t_0$:
  1. Dynamic hedging
  2. Evaluation of the τ-year ahead Value-at-Risk

- To avoid the need for nested simulations, the 'approximation of survival functions' method is used.
  - $\mathrm{E}_t^{(Q)}[\mathcal{S}_{x_0+t-t_0,t}(s)]$ depends on $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$.
  - A Taylor's approximation is applied to (the probit transformed) $\mathrm{E}_t^{(Q)}[\mathcal{S}_{x_0+t-t_0,t}(s)]$ around the best estimates of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ as of $t = t_0$.
  - The absolute percentage errors are less than 0.03%.

Moneyness

For K-options with a strike value of K and maturity date of T, the following moneyness metric is defined:

$$\text{Moneyness} = \frac{K - \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\mathrm{Var}_t^{(Q)}[\kappa_T^{(i)}]}}.$$

For any $t_0 \le t < T$, we have the following:

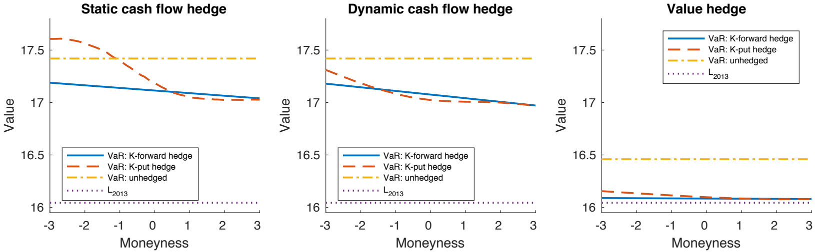
- If $K = \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]$, the K-options are called at-the-money.
- If $K < \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]$, then the K-call is called in-the-money while the K-put is called out-of-the-money.
- If $K > \mathrm{E}_t^{(Q)}[\kappa_T^{(i)}]$, then the K-put is called in-the-money while the K-call is called out-of-the-money.

Baseline Assumptions

- The annuity liability is sold to individuals aged $x_0 = 65$ at time t_0. The limiting age is assumed to be $\omega = 100$.
- The composition of the hedge portfolio is either
  - one K-forward (written as a fixed-rate receiver) on each of the two CBD mortality indexes, or
  - one K-put on each of the two CBD mortality indexes.

  The 'Moneyness' of the two instruments used are identical.
- $T = 15$, $\boldsymbol{\lambda} = (0.175, 0.175)'$, and $r_f = 0.02$.
- Both the CBD mortality indexes and the annuity liability are linked to the mortality experience of EW males.

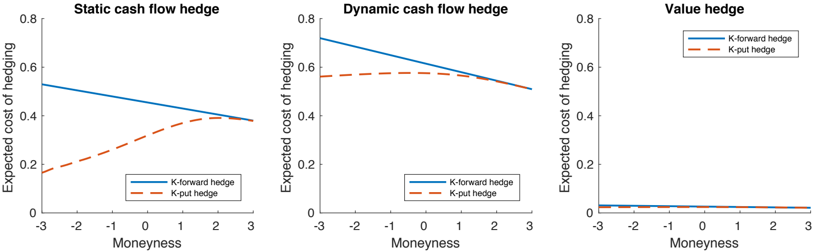
Baseline Results



K-put vs. K-forward

Factor (1): Cost of hedging

- A K-put hedge is less costly
- Reason: only the downside risk has to be 'paid'

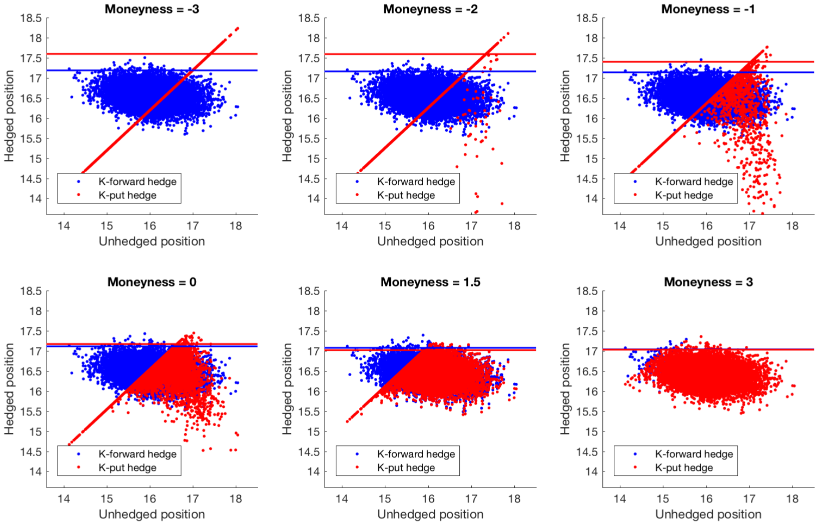


K-put vs. K-forward

Factor (2): Effectiveness as a hedging instrument

- The delta of a K-put is always smaller in magnitude than that of the corresponding K-forward.
- As such, the hedge ratio (the notional amount) of a K-put hedge is always larger in magnitude than that of the corresponding K-forward hedge.
- Compared to a K-forward hedge, a K-put hedge is more effective in the sense that it will pay a larger payoff to the hedger in an adverse scenario.

K-put vs. K-forward



Other Interesting Findings

- For cash flow hedges, a K-put hedge tends to yield a lower VaR compared to the corresponding K-forward hedge when the market prices of risk are high.
- However, if the market prices of risk are too high, both K-put and K-forward hedges may no longer be economically justifiable.
- For static cash flow hedges, a K-put hedge tends to result in a smaller VaR compared to the corresponding K-forward hedge when the times-to-maturity of the hedging instruments are long.
- Value hedges are highly effective, even when the market prices of risk are high.

Outline

Conclusion

Security structures:

- Introduced K-options written on the CBD mortality indexes
- Explained how K-options may be utilized by hedgers

Pricing:

- Derived exact analytical pricing formulas for K-forwards and options

Hedging:

- Developed three types of longevity hedges
- Derived analytical expressions for the longevity Greeks of K-forwards and options
- Examined the relative performance between K-option hedges and K-forward hedges in different circumstances